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Collision Integrals for Attractive Potentials

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Motivated by previous discussions of particle interactions under the Manev potential $U(r) = -\alpha/r - \varepsilon/r^2$, we construct the collision integrals for attractive potentials $U(r)$ satisfying the condition $U(r)\, r^2 \to -\varepsilon$ as $r \to 0$ with $\varepsilon \geqslant 0$. For $\varepsilon = 0$, we obtain a Boltzmann-type integral with a collision law allowing "spiral" interactions and nonunique correspondence between impact parameter and scattering angle. For $\varepsilon > 0$, an additional Smoluchowski-type coagulation integral arises. All these integrals are derived and possible applications are discussed.

KEY WORDS: Attractive power potential; scattering angle; collision integral; coagulation integral.

INTRODUCTION

In refs. 2 and 3, we investigated a Vlasov equation for the potential

$$U(r) = -\frac{\alpha}{r} - \frac{\varepsilon}{r^2}, \qquad \alpha, \varepsilon > 0 \tag{0.1}$$

We referred to this Vlasov equation as "Vlasov–Manev equation," after the Bulgarian physicist Manev who studied this singluar type of correction to the Coulomb potential in the 1920s. The properties of this Vlasov–Manev equation, the local well-posedness of the initial value problem and the general nonexistence of global solutions were the topic of the above mentioned references. We mention that potentials like (0.1) were studied by others long before Manev, most notably by Newton in the Principia.

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Consider now a mollification of the potential U, i.e., let

$$U_\delta(r) = U * \omega_\delta$$

with $\omega_\delta(r) = (2\pi\delta)^{-3/2}\, e^{-r^2/2\delta}$. For $\varepsilon \ll \delta$, one would expect the mollification to suppress the effects of the ε/r^2 part of the potential relative to the α/r part—the classical Vlasov equation (given by mean field effects) with a small smooth perturbation in the potential would emerge. On the other hand, if $\delta = 0$, i.e., without any mollification of the potential, one should also take pair collisions into account, i.e., the Vlasov–Manev equation should be complemented by collision and (as we will show in this paper) coagulation terms. For $\varepsilon = 0$ in (0.1), it is well-known that this leads to the Landau–Fokker–Planck collision term, which is small near the Vlasov limit (see ref. 4). A qualitative effect of pair collisions for $\varepsilon = 0$ is a "very slow" relaxation to the local Maxwellian. Moreover, the collision term for $\varepsilon = 0$ does not depend on the sign of α in (0.1), i.e., there is no difference between repulsive (say, electron-electron collisions in a plasma) and attractive forces in the collision integral.

The objective of this paper is to describe the effects of pair collisions for $\varepsilon > 0$ in (0.1). We consider a more general context, namely, how to construct the Boltzmann collision integrals for non-Coulomb attractive forces. As we-will see, the collision integral does actually not make sense in some situations; e.g., certain types of collisions for the potential (0.1) will result in coagulation of two colliding particles, and such effects should be accounted for. Here, coagulation is understood in the sense that point particles may, on sets in phase space of positive measure, find themselves at the exact same spot after finite time. For details, see Sections 2 and 3.

The paper is organized as follows. In Section 1, we consider the scattering problem for an attractive potential $U(r)$ such that $\lim_{r\to 0} |U(r)|\, r^2 < \infty$. Two new effects occur in this situation: (1) The possibility of "spiral" trajectories and non-unique correspondence between impact parameter and scattering angle, and (2) coagulation for $|U(r)|\, r^2 \to \text{const.}$ as $r \to 0$. In Section 2 we construct the Boltzmann collision integral for elastic attractive scattering, while Section 3 contains the construction of the Smoluchowski coagulation integral for the case of coagulation.

1. THE SCATTERING PROBLEM

We begin by recalling well-known facts from classical mechanics (see, e.g., ref. 5). Consider the motion of a particle with mass m in the central

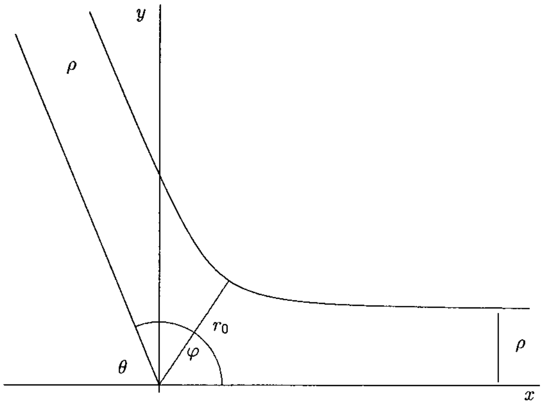

Fig. 1. Scattering with a repulsive potential.

field $U(r)$ such that $U \in C^1(0, \infty)$ and $U(r) \to 0$ as $r \to \infty$. We assume that the equation

$$U(r) + \frac{M^2}{2mr^2} = E \tag{1.1}$$

has exactly one positive solution $r = r_0$ for any two positive constants M (angular momentum) and E (energy).

In this section, we will emphasize vectors by using the classical arrow superscripts to avoid notational problems. These arrows will be omitted in Sections 2 and 3.

Let $\vec{u}_\pm$ denote the velocities of a particle for $t = \pm\infty$ (t is time), and $M = m\rho\,|\vec{u}_\pm|$ be the absolute value of angular momentum. The number $\rho \geqslant 0$ is called an impact parameter. It is always possible to construct a plane P with Cartesian cooordinates (x, y) such that $\vec{u}_- = \{-u, 0\}$, $\vec{u}_+ = \{u'_x, u'_y\}$, where $u'^2_x + u'^2_y = u^2$. A potential is called "repulsive" if $U'(r) < 0$ and "attractive" if $U'(r) > 0$. A typical particle trajectory in the plane P is shown in Fig. 1.

For this trajectory, $M_z = \rho u > 0$ (the generalization to the case $M_z = -\rho u < 0$ follows immediately by symmetry). We set $\vec{u}_+ = u(\cos\varphi, \sin\varphi)$ (see Fig. 1) and after standard calculations[4, 5] obtain an integral formula

$$\varphi = 2\rho \int_{r_0}^{\infty} \frac{dr}{r^2\sqrt{1 - (2U(r)/mu^2) - (\rho/r)^2}} \tag{1.2}$$

with $(2U(r_0)/mu^2)+(\rho^2/r_0^2)=1$ (from (1.1)). The identity (1.2) defines $\varphi\in[0,\pi]$ as a function of ρ for fixed energy $E=mu^2/2$. Under simple and natural conditions on the (repulsive) potential $U(r)$ the function $\varphi(\rho)$ increases monotonically from $\varphi(0)=0$ to $\varphi(\infty)=\pi$, such that the inverse function $\rho=\rho(\varphi)$ exists. In the repulsive case, the scattering angle $0<\theta<\pi$ between $\vec{u}_-$ and $\vec{u}_+$ is given as $\theta=\pi-\varphi$. Let $\tilde{\rho}(\theta)=\rho(\pi-\theta)$, then we define a differential cross-section $\sigma(u,\theta)$ by

$$\sigma(u,\theta)=\frac{\tilde{\rho}(\theta)}{\sin\theta}\left|\frac{d}{d\theta}\tilde{\rho}(\theta)\right| \tag{1.3}$$

The function $|u|\,\sigma(u,\theta)$ is the appropriate kernel for the Boltzmann collision integral associated with the potential $U(r)$ under consideration.[4] In two dimensions, the function

$$\sigma_2(u,\theta)=\frac{d}{d\theta}\tilde{\rho}(\theta) \tag{1.4}$$

plays the corresponding role.

To proceed, we have to make more specific assumptions on the potential U. We assume that U is such that

(A) $U(r)\in C^1(0,\infty)$, $U'(r)>0$, $U(r)\to 0$ as $r\to\infty$;

(B) $(d/dr)(r^2U(r))\leqslant 0$, $r^2U(r)\to -\varepsilon$ as $r\to 0$, with $\varepsilon\geqslant 0$.

Lemma. Let $\Psi(r)=(2U(r)/mu^2)+(\rho^2/2r^2)$. If (A) and (B) hold, then the equation

$$\Psi(r_0)=1$$

has a unique solution $r_0>0$ for any $\rho>\rho_*$, where $\rho_*=(2\varepsilon/mu^2)^{1/2}$. There are no solutions for $0<\rho<\rho_*$.

Proof. Let $\rho>\rho_*$. Then $\Psi(r)\to\infty$ as $r\to 0$ and $\Psi(r)\to 0$ as $r\to\infty$. It is thus sufficient to prove that $\Psi'<0$ while $\Psi(r)>0$. To this end, assume the opposite, i.e., that there is a $r_1>0$ with $\Psi(r_1)>0$ and $\Psi'(r_1)=0$. Then

$$\frac{\rho^2}{r_1^2}=\frac{r_1U'(r_1)}{mu^2}$$

and therefore

$$u\Psi(r_1)=\frac{1}{mur_1}\,(r^2U(r))'\,|_{r=r_1}$$

Fig. 2. Scattering with an attractive potential.

The right hand side is nonpositive because of (B), contradicting $\Psi(r_1)>0$. This proves the first part of the assertion (for $\rho>\rho_*$, $\varepsilon\geqslant 0$). For $\rho<\rho_*$ with $\varepsilon>0$ (B) implies that $U(r)\leqslant -\varepsilon/r^2$, and it follows that $\Psi(r)\leqslant 0$ for all $r>0$ in this case.

In the sequel we shall always assume that (A) and (B) are satisfied. The integral formula remains valid, but the correspondence between ρ and φ is not unique anymore. Typical trajectories for this situation are shown in Fig. 2 (the scales are different in the two pictures; the collision parameter ρ is smaller in the second case).

Intuitively, one expects in this situation that $\varphi'(\rho)<0$, $\varphi(\infty)=\pi$, and that there is a critical collision parameter ρ_c, positive or zero, such that $\lim_{\rho\searrow\rho_c}\varphi(\rho)=\varphi_{\max}>\pi$. $\varphi_{\max}$ can be infinite.

Let us now consider a power-potential

$$U(r)=-\frac{\alpha}{r^\gamma},\qquad \alpha>0,\qquad 1\leqslant\gamma<2 \tag{1.5}$$

Formula (1.2) becomes

$$\varphi=2\rho\int_{r_0}^{\infty}\frac{dr}{r^2\sqrt{1+(s/r)^\gamma-(\rho/r)^2}},\qquad \text{with}\quad s^\gamma=\frac{2\alpha}{mu^2} \tag{1.6}$$

or, from (1.2),

$$\varphi=\varphi(z)=2\int_0^{x_0}\frac{dx}{\sqrt{1+zx^\gamma-x^2}} \tag{1.7}$$

with $z=2\alpha/mu^2\rho^\gamma$ and $1+zx_0^\gamma-x_0^2=0$.

Let $\gamma=2-1/n$, $n=1, 2,\ldots$, then the substitution $x=(zy)^n$ leads to

$$\varphi(z)=2n\int_0^{y_0}dy[z^{-2n}y^{2-2n}+y(1-y)]^{-1/2} \tag{1.8}$$

with $y_0(1-y_0)+y_0^{2-2n}z^{-2n}=0$.

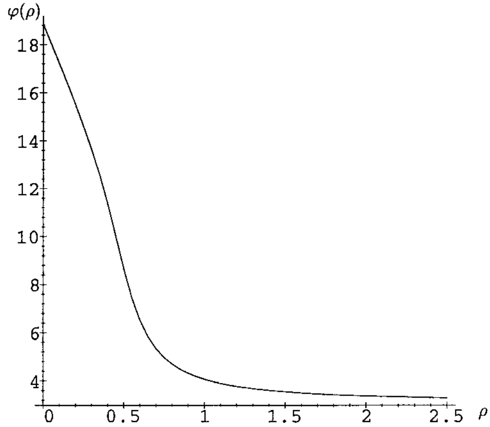

Fig. 3.

If $z \to \infty$, i.e., $\rho \to 0$, then

$$\varphi(z) \to \varphi_{\max} = 2n \int_0^1 \frac{dy}{\sqrt{y(1-y)}} = 2n\pi \tag{1.9}$$

Hence simple trajectories (i.e., $\pi < \varphi < 2\pi$) occur only for $n = 1$ (the Coulomb or Newtonian potential), while in the general case $n = 2, 3,...$ we get spiral trajectories with $(n-1)$ loops. The dependence $\varphi(\rho)$ for $n = 3$ is depicted in Fig. 3. This figure, as all others in this paper, were produced with the MAPLE symbolic computation package.

To compute the scattering angle $\theta \in [0, \pi]$ between the velocities $\vec{u}_-$ and $\vec{u}_+$, we observe the following elementary geometric relationship between $\varphi \in (\pi, \infty)$ and $\theta \in [0, \pi]$ (see Fig. 2, where $k = 1$):

$$\begin{aligned} &\text{if} \quad (2k-1)\,\pi \leqslant \varphi \leqslant 2k\pi, \qquad \text{then} \quad \theta = \varphi - (2k-1)\,\pi, \qquad \text{and} \\ &\text{if} \quad 2k\pi \leqslant \varphi \leqslant (2k+1)\,\pi, \qquad \text{then} \quad \theta = (2k+1)\,\pi - \varphi \end{aligned} \tag{1.10}$$

For attractive potentials $U(r)$ such that $r^2\,|U(r)| \to 0$ and $r \to 0$, the formulas (1.2) and (1.10) define the scattering angle θ uniquely as a function of the impact parameter $\rho \in (0, \infty)$.

We next consider the scattering problem for the potential

$$U(r) = -\frac{\varepsilon}{r^2}, \qquad \varepsilon > 0 \tag{1.11}$$

For this case, the integral (1.2) gives the simple explicit formula

$$\varphi = \pi \left[1 - \left(\frac{\rho_*}{\rho} \right)^2 \right]^{-1/2}, \qquad \text{with} \quad \rho_*^2 = \frac{2\varepsilon}{mu^2} \tag{1.12}$$

for $\rho > \rho_*$. Note that $\varphi \to \infty$ as ρ decreases from ∞ to ρ_*.

If $\rho < \rho_*$, then the particle "falls into the center $r = 0$." To analyse this further, observe that the scattering problem appears initially from a reduction of the two body problem. Consider two particles with masses m_i, positions $\vec{x}_i$ and velocities $\vec{v}_i$ $(i = 1, 2)$, interacting via the potential $U(|\vec{x}_1 - \vec{x}_2|)$. A standard transformation to the center-mass frame of reference,

$$\begin{aligned} \vec{X} &= \frac{m_1 \vec{x}_1 + m_2 \vec{x}_2}{m_1 + m_2}, \qquad \vec{x} = \vec{x}_1 - \vec{x}_2 \\ \vec{V} &= \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2}, \qquad \vec{u} = \vec{v}_1 - \vec{v}_2 \end{aligned} \tag{1.13}$$

reduces the two-body problem to the problem of one body (with mass $m = (m_1 m_2)/(m_1 + m_2)$, position $\vec{x}$ and velocity $\vec{u}$) in the central field $U(|\vec{x}|)$. It is well known[5] that for the potential (1.11) a global (in time) solution of the two-body problem does not exist for sufficiently small $(\rho < \rho_*)$ relative angular momentum. For given initial conditions

$$\begin{gathered} \vec{x}(0) = \vec{x}_0, \qquad \vec{u}(0) = \vec{u}_0, \qquad \vec{x}_0 \cdot \vec{u}_0 < 0, \qquad E = \frac{m\vec{u}_0^{\,2}}{2} - \frac{\varepsilon}{|\vec{x}_0|^2} < 0 \\ M^2 = m^2 [x_0^2 u_0^2 - (\vec{x}_0 \cdot \vec{u}_0)^2] < 2\varepsilon m \end{gathered} \tag{1.14}$$

there exists a time instant $t_0 \in (0, \infty)$ such that

$$\vec{x}(t) \to 0, \qquad |\vec{u}(t)| \to \infty \qquad \text{as} \quad t \nearrow t_0$$

(this is best seen by noting that E is the time-invariant energy for the system in the $\vec{x}$, $\vec{u}$ coordinates, and that $(d^2/dt^2)(m\vec{x}^2) = 4E < 0$). The question arises how the solution of the two-body problem should be continued for $t > t_0$. A natural way to do so is to assume that the two particles simply coagulate at $t = t_0$, i.e., they form a heavier particle with mass $M = m_1 + m_2$ and velocity $\vec{V}$. This guarantees momentum conservation, while there is in general a loss of energy (the kinetic energy of the "new" particle is smaller than the total mechanical energy (kinetic plus potential) of the two particles at $t = 0$). This corresponds to the implicit assumption that the new heavier particle possesses internal energy which, however, is irrelevant for our purposes.

We emphasize that this coagulation concept destroys the conservative character of the system, as energy is not conserved. We also hasten to add that the coagulation process is different from the type of coagulation encountered in chemical separations and recombinations; indeed, our particles are to be thought of as dimensionless "black holes" which interact under the strongly singular attractive potential under consideration.

Summarizing, we describe collisions between two particles with masses m_i, $i=1,2$ and velocities $\vec{v}_i$, $i=1,2$ before the collision and interacting with the potential (1.11) as follows. If the impact parameter ρ satisfies $\rho>\rho_*$ (see (1.42)), then an elastic scattering with scattering angle $\theta\in[0,\pi]$ given by (1.10) and (1.12) occurs. If $\rho<\rho_*$ we have a coagulation, i.e., the collision results in the formation of a heavier particle with mass $M_+=m_1+m_2$ and velocity $\vec{V}=(m_1\vec{v}_1+m_2\vec{v}_2)/M_+$.

This analysis of the pair collision process applies not only to the potential (1.11), but also to other potentials $U(r)$ for which $U(r)\,r^2\to-\varepsilon$ as $r\to0$. The only difference is that instead of the simple formula (1.12) one has to employ the general integral formula (1.2) to describe elastic scattering for $\rho>\rho_*$.

If $\rho<\rho_*$, then particles coagulate in the same way independently of the specific form of the potential; only the behavior for $r\to0$ matters.

In particular, for the potential

$$U(r)=-\frac{\alpha}{r}-\frac{\varepsilon}{r^2} \tag{1.15}$$

with $\alpha,\varepsilon>0$, the integral (1.2) can be calculated in explicit form as

$$\varphi=\left[\pi+\arctan\frac{\alpha}{mu^2\sqrt{\rho^2-\rho_*^2}}\right]\left[1-\left(\frac{\rho_*}{\rho}\right)^2\right]^{-1/2} \tag{1.16}$$

where $\rho>\rho_*$ and $\rho_*^2=2\varepsilon/mu^2$.

The purpose of this section was to collect all necessary calculations for the scattering problem. We are now ready for the construction of collision integrals for attractive potentials.

2. COLLISION INTEGRALS FOR ELASTIC SCATTERING

To simplify our notation, we now denote vectors without the arrow superscript.

Consider a repulsive pair potential defining a scattering cross-section as discussed in Section 1. If $f = f(x, v, t)$ is the particle density of a rarefied gas, the collision term for the Boltzmann equation is

$$\int_{\mathbb{R}^3 \times S^2} dw\, dn\, \sigma(|u|, \phi)\, |u|\, [f(v')\, f(w') - f(v)\, f(w)] \tag{2.1}$$

where $u = v - w$, the arguments x and t have been suppressed, and the post-collisional velocities v' and w' are given by

$$v' = \tfrac{1}{2}(v + w + |u|\, n), \qquad w' = \tfrac{1}{2}(v + w - |u|\, n)$$

Here, $n = (\sin\theta \cos\alpha, \sin\theta \sin\alpha, \cos\theta)$, $dn = \sin\theta\, d\theta\, d\alpha$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \alpha \leqslant 2\pi$.

We now generalise the collision integral (2.1) to the case of attractive potentials. To start, we write (2.1) in the more compressed form

$$I(f, f)(v) = \int_{\mathbb{R}^3} dw\, |v - w|\, \Psi\left(\frac{v + w}{2}, v - w\right) \tag{2.2}$$

with

$$\Psi(V, U) = \int_{S^2} dn\, \sigma(|U|, \sigma) \{F(V, |U|\, n) - F(V, U)\} \tag{2.3}$$

and $F(V, U) = f(V + (U/2))\, f(V - (U/2))$.

Note that the original form of the Boltzmann collision integral includes an integration over the impact parameter $\rho \in (0, \infty)$, and *not* over the scattering angle θ. With the integral over ρ and (1.3), the integral in (2.3) becomes (we omit the argument V)

$$\Psi(U) = \int_0^\infty d\rho\, \rho \int_0^{2\pi} d\alpha \{F(|U|\, n) - F(U)\} \tag{2.4}$$

where $n \in S^2$ is still defined as before, but a polar scattering angle θ is now a function of ρ as defined by the scattering problem which we discussed in Section 1. If the intermolecular potential $U(r)$ is positive and monotonically decreasing with r, we have a one-to-one correspondence between $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \rho < \infty$, and (1.3) defines the relationship between (2.3) and (2.4).

In the sequel we consider the integral (2.3) for attractive potentials when the dependence between ρ and θ is given by the formulas (1.2) and (1.10). We assume monotone dependence between ρ and φ, as indicated in Fig. 3.

To simplify our notation, we write the integral (2.4) in the compressed form

$$\Psi(U) = \int_0^{\infty} d\rho\, \rho G[\theta(\rho)] \tag{2.5}$$

with $G[\theta(\rho)] = \int_0^{2\pi} d\alpha \{F(|U|\, n) - F(U)\}$, where $n = (\theta(\rho), \alpha)$ is given in spherical coordinates with the polar axis in direction U. Using (1.2), we define ρ_n, $n = 1, 2,...$, by $\rho_1 = \infty$ and $\varphi(\rho_n) = n\pi$, $n = 2, 3,...$. If there is a $\varphi_{\max}$ such that $\varphi(\rho) \to \varphi_{\max}$ as $\rho \to 0$, and if we set $\varphi_{\max} = N\pi + \varphi_0$ with $0 \leqslant \varphi_0 \leqslant \pi$, we set $\rho_n \equiv 0$ for $n > N$. Using (1.10), we rewrite (2.4) as

$$\begin{aligned}\Psi(U) = \sum_{k=1}^{\infty} \Bigg\{ &\int_{\rho_{2k}}^{\rho_{2k-1}} d\rho\, \rho G[\varphi(\rho) - (2k-1)\,\pi] \\ &+ \int_{\rho_{2k+1}}^{\rho_{2k}} d\rho\, \rho G[(2k+1)\,\pi - \varphi(\rho)] \Bigg\}\end{aligned} \tag{2.6}$$

$\varphi(\rho)$ being defined in (1.2).

Let

$$\begin{aligned}\tilde{\rho}^{(2k)}(\theta) &= \rho[(2k-1)\,\pi + \theta], \qquad k = 1, 2,... \\ \tilde{\rho}^{(2k+1)}(\theta) &= \rho[(2k+1)\,\pi - \theta], \qquad k = 1, 2,...\end{aligned} \tag{2.7}$$

$0 \leqslant \theta \leqslant \pi$. We remark that the inverse function $\rho(\varphi)$ exists in view of our assumption that $\varphi(\rho)$ is strictly monotone.

It is now natural to introduce a set of "partial" differential cross sections (see (1.3) for comparison)

$$\sigma_n(\theta) = \frac{\tilde{\rho}^{(n)}(\theta)}{\sin\theta} \left| \frac{d}{d\theta} \tilde{\rho}^{(n)}(\theta) \right|, \qquad n = 2, 3,... \tag{2.8}$$

and to set $\rho = \tilde{\rho}^{(2k)}(\theta)$ or $\rho = \tilde{\rho}^{(2k+1)}(\theta)$ in each of the integrals in (2.6). Returning to the integration variable $\theta \in [0, \pi]$, the result is

$$\Psi(U) = \int_0^{\pi} d\theta \sin\theta F(\theta)\, \hat{\sigma}(|U|, \theta) \tag{2.9}$$

with

$$\hat{\sigma}(|U|, \theta) = \sum_{n=2}^{\infty} \sigma_n(|U|, \theta) \tag{2.10}$$

If $\varphi_{\max} < \infty$, the sum in (2.10) is actually a finite sum. For example, $\sigma_n(|U|, \theta) = 0$ for $n = 3, 4,...$ for the Newtonian potential (1.5) with $\gamma = 1$, while $\sigma_2(|U|, \theta)$ is merely the classical Rutherford cross-section in this case (the integral (2.5) is then actually divergent, a fact we ignore for the current formal discussion).

The partial differential cross-sections $\sigma_n(\theta)$ in (2.8) allow the following physical interpretation: Each function $\sigma_n(\theta)$ is associated with the relative contribution of those trajectories which have exactly $n-1$ intersections with a real axis directed along U.

To summarize we obtain for an attractive potential $U(r)$ the usual Boltzmann collision integral (2.1), provided that $|U(r)|\, r^2 \to 0$ as $r \to 0$. The only difference is that we have to use a generalized differential cross-section $\hat{\sigma}(|U|, \theta)$ as given by (2.10) to replace the usual cross-section $\sigma(|U|, \theta)$.

We now discuss the important case when $|U(r)|\, r^2 \to \varepsilon > 0$ as $r \to 0$ beginning with the simplest case (1.11). Elastic scattering is again described by the Boltzmann collision integral in the form (2.2)–(2.4), provided that we integrate over $d\rho$ in (2.4) *not* from 0 to ∞, but from $\rho = \rho_{\min} = \rho_*$ (as given by (1.12)) to ∞. All the above arguments carry over. Inverting the explicit formula (1.12), we find

$$\rho^2 = \rho_*^2 [1 - (\pi/\varphi)^2]^{-1} \tag{2.11}$$

hence

$$\rho \frac{d\rho}{d\varphi} = -\frac{\rho_*^2 \pi^2}{\varphi^3} [1 - (\pi/\varphi)^2]^{-2} = -\frac{(\pi \rho_*)^2 \varphi}{[\varphi^2 - \pi^2]^2} \tag{2.12}$$

From (2.7), (2.8)

$$\begin{aligned} \sin\theta\, \sigma_{2k}(\theta) &= (\pi\rho_*)^2 \frac{(2k-1)\pi + \theta}{\{[(2k-1)\pi + \theta]^2 - \pi^2\}^2} \\ \sin\theta\, \sigma_{2k+1}(\theta) &= (\pi\rho_*)^2 \frac{(2k+1)\pi - \theta}{\{[(2k+1)\pi - \theta]^2 - \pi^2\}^2} \end{aligned} \tag{2.13}$$

$k = 1, 2,...$. The formulas (2.10) and (2.13) (note that the convergence in (2.10) is straightforward) define a generalized elastic cross-section $\hat{\sigma}(|U|, \theta)$ for the potential $U(r) = -\varepsilon/r^2$, $\varepsilon > 0$. We also observe that $\sigma_2(\theta)$ diverges as $\theta \to 0$.

The potential given by (1.15) can be treated in much the same way. The only difference in the construction of $\hat{\sigma}(|U|, \theta)$ is that one has to substitute (1.16) for (1.12) in the analysis. As (1.16) cannot be inverted explicitly, one has to use numerical analysis to compute $\hat{\sigma}$ in this situation. No conceptual difficulties arise; in particular, convergence of the series (2.10) holds because the second term in (1.15) dominates as $n \to \infty$.

We end this section with two remarks.

Remark 2.1. The Boltzmann collision integral with cross-section (2.10) readily generalizes to the two-dimensional situation. To this end, all one has to do is substitute the formula (1.4) for (1.3) and repeat the above calculations.

Remark 2.2. Another generalization applies to the situation where particles have different masses. Let $f_i(x, v, t)$ be the densities associated with particles with mass m_i, $i = 1, 2$. The Boltzmann equation for f_1 will read

$$\partial_t f_1 + v \cdot \nabla_x f_1 = I(f_1, f_1) + I(f_1, f_2) \tag{2.14}$$

We have already discussed the first collision term in (2.14). As for the second one, it can be rewritten analogously to (2.2):

$$I(f_1, f_2) = \int dw \, |v - w| \, \Psi\left(\frac{m_1 v + m_2 w}{m_1 + m_2}, v - w\right) \tag{2.15}$$

where $\Psi(\cdots)$ is defined as in (2.3) with

$$F(V, U) = f_1\left(V + \frac{m}{m_1} U\right) f_2\left(V - \frac{m}{m_2} U\right) \tag{2.16}$$

and $m = m_1 m_2/(m_1 + m_2)$ is the reduced mass. This reduced mass also has to be used in (1.2). All other calculations apply without changes.

3. BOLTZMANN–SMOLUCHOWSKI COLLISION INTEGRALS

We again consider first the potential

$$U(r) = -\frac{\varepsilon}{r^2}, \qquad \varepsilon > 0 \tag{3.1}$$

As discussed at the end of Section 1, a classical solution of the scattering problem for this potential exists only for sufficiently large impact parameters $\rho^2 > \rho_*^2 = 2\varepsilon/\mu u^2$ and reduced mass $\mu = (m_1 m_2)/(m_1 + m_2)$. If $\rho < \rho_*$, we have to consider a generalized solution of the scattering problem which incorporates a coagulation process. A consequence is that we have to consider particles with different masses in any case, because particle masses change in the collision (coagulation) process. It therefore becomes necessary to consider particle masses as an additional independent variable and derive kinetic equations for a distribution density $f(m, x, v, t)$, where $m > 0$ denotes the particle mass.

The total particle mass is then given by the formula

$$M = \iint dx\, dv \int_0^\infty dm\, f(m, x, v, t) \tag{3.2}$$

We assume that a pair collision with an impact parameter $\rho < \rho_*$ results in coagulation of particles, i.e., in the "reaction"

$$(m_1, v_1) + (m_2, v_2) \to (M, V) \tag{3.3}$$

with $M = m_1 + m_2$, $V = (m_1 v_1 + m_2 v_2)/M$. Note that the postcollisional velocity V does not depend on the impact parameter ρ provided that $\rho < \rho_*$. The coagulation process is therefore completely determined by a total coagulation cross-section

$$\sigma_c(u; m_1, m_2) = \pi \rho_*^2 \tag{3.4}$$

The time which is needed for a typical scattering or coagulation event is assumed to be negligible with respect to the macroscopic time scale.

We have to distinguish the following two possible cases:

(1) The pair potential $U(r)$ (e.g., the constant ε in (3.1)) does not depend on the masses m_1 and m_2 of the colliding particles

(2) Or, there is such a dependence. We then write $U(r; m_1, m_2)$.

In the sequel we consider the more general situation (2). An elastic differential cross-section can then be calculated in the same way as described in Section 2, by using the mass-dependent potential $U(t; m_1, m_2)$. For example, let us set $\varepsilon = \varepsilon(m_1, m_2)$ in (3.1). We then set

$$\rho_*^2 = \frac{2\varepsilon(m_1, m_2)(m_1 + m_2)}{m_1 m_2 u^2} \tag{3.5}$$

in (2.13) and calculate a differential cross-section $\hat{\sigma}(u, \theta; m_1, m_2)$ by the formulas (2.10)–(2.13). A change similar to (3.5) has to be made in (3.4) for a total coagulation cross-section $\sigma_c(u; m_1, m_2)$.

If we do not consider the simple potential given in (3.1), but a more general attractive potential $U(r; m_1, m_2)$ such that

$$|U(r; m_1, m_2)|\, r^2 \to \varepsilon(m_1, m_2) \tag{3.6}$$

as $r \to 0$, then no changes arise in (3.3)–(3.5) (the coagulation condition remains the same), while all details of calculating the elastic cross-section $\hat{\sigma}(u, \theta; m_1, m_2)$ have already been presented above.

We are now ready to write the Boltzmann–Smoluchowski equation for a distribution density $f(m, x, v, t)$:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = I_B + I_{Sm} \tag{3.7}$$

where I_B and I_{Sm} denote the Boltzmann and Smoluchowski collision integrals, respectively. We assume that particles interact via an attractive pair potential $U(r; m_1, m_2)$ satisfying the condition (3.6). First calculate cross-sections $\hat{\sigma}(u, \theta; m_1, m_2)$ and $\sigma_c(u; m_1, m_2)$ as described earlier. Then

$$\begin{aligned} I_B = \int_0^\infty dm_1 \int_{\mathbb{R}^3 \times S^2} dw\, dn\, |U|\, \hat{\sigma}(|U|, \theta; m, m_1) \\ \times \{ f(m, v')\, f(m_1, w') - f(m, v)\, f(m_1, w) \} \end{aligned} \tag{3.8}$$

with $U = v - w$, $U \cdot n = |U| \cos\theta$, $v' = V + (\mu/m)\, U \cdot n$, $w = V - (\mu/m)\, U \cdot n$, $V = (mv + m_1 w)/(m + m_1)$, $\mu = (m_1 m)/(m_1 + m)$. This is the obvious generalization of the usual Boltzmann collision integral for particles with different masses.

The Smoluchowski collision integral is

$$\begin{aligned} I_{Sm} = \frac{1}{2} \int_0^\infty dm_1 \int_0^\infty dm_2 \int_{\mathbb{R}^3 \times \mathbb{R}^3} dv_1\, dv_2\, |U|\, \sigma_c(|U|; m_1, m_2) \\ \times f(m_1, v_1)\, f(m_2, v_2)\, \delta(m_1 + m_2 - m)\, \delta\left(\frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} - v \right) \\ - f(m, v) \int_0^\infty dm_1 \int dw\, f(m_1, w)\, |v - w|\, \sigma_c(|v - w|; m_1, m_2) \end{aligned} \tag{3.9}$$

with $U = v_1 - v_2$. The factor 1/2 in front of the first integral gives the correct number of inelastic collisions. We have omitted the arguments x and t throughout.

The potential $U(r; m_1, m_2)$ and the cross sections $\hat{\sigma}(|U|, \theta; m_1, m_2)$ are symmetric functions of the masses m_1 and m_2. The first term on the right of (3.9) can be simplified further, but the present form is advantageous for the calculation of inner products

$$(\psi, I_{Sm}) = \int_0^\infty dm \int_{\mathbb{R}^3} dv\, \psi(m, v)\, I_{Sm}(m, v)$$

with test functions ψ. After some elementary transformations, one finds

$$\begin{aligned}(\psi, I_{Sm}) = \frac{1}{2} \int dm_1 \int dm_2 \int_{\mathbb{R}^3 \times \mathbb{R}^3} & dv_1\, dv_2\, f(m_1, v_1)\, f(m_2, v_2) \\ \times |U|\, \sigma_c(|U|; m_1, m_2) & \left[\psi\left(m_1 + m_2, \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}\right) \right. \\ & \left. - \psi(m_1, v_1) - \psi(m_2, v_2) \right]\end{aligned}$$

$U = v_1 - v_2$, from which conservation of mass and momentum readily follow:

$$(m, I_{Sm}) = 0 = (mv, I_{Sm})$$

The dependence of the Smoluchowski integral on the relative speed $|U|$ has universal character as follows from the formulas (3.4)–(3.5), provided that condition (3.6) holds.

We conclude by presenting explicit formulas for the simplest case (3.1), assuming that ε in (3.1) depends on the masses. Let (see (2.10–(2.13))

$$\begin{aligned} g(\theta; m_1, m_2) &= 2\, \frac{\pi^2 \varepsilon(m_1, m_2)(m_1 + m_2)}{m_1 m_2}\, g(\theta) \\ \sin\theta\, g(\theta) &= \sum_{k=1}^{\infty} \left\{ \frac{(2k-1)\,\pi + \theta}{\{[(2k-1)\,\pi + \theta]^2 - \pi\}^2} + \frac{(2k+1)\,\pi - \theta}{\{[(2k+1)\,\pi - \theta]^2 - \pi\}^2} \right\} \\ p(m_1, m_2) &= 2\, \frac{\pi \varepsilon(m_1, m_2)(m_1 + m_2)}{m_1 m_2} \end{aligned} \tag{3.10}$$

The collision terms in (3.7) then become

$$\begin{aligned} I_B = \int_0^\infty dm_1 \int_{\mathbb{R}^3 \times \mathbb{R}^3} & dw\, dn\, \frac{g(\theta; m_1, m_2)}{|v - w|} \\ & \times \{ f(m, v')\, f(m_1, w') - f(m, v)\, f(m_1, w) \} \end{aligned} \tag{3.11}$$

and

$$I_{Sm}=\frac{1}{2}\int_0^m ds\int_{\mathbb{R}^3} du\,\frac{p(s,m-s)}{|U|}\,f\left(s,v+\frac{s}{m}U\right)f\left(m-s,v-\frac{m-s}{m}U\right)$$

$$-f(m,v)\int_0^\infty ds\int dw\,\frac{p(m,s)}{|v-w|}\,f(s,w) \tag{3.12}$$

where v' and w' are defined after (3.8).

Remarkably, in the two-dimensional case ($x, v\in\mathbb{R}^2$) the potential (3.1) results in a collision integral of Maxwell type (i.e., the collision kernel (cross-section) becomes independent of the relative speed), which can be significantly simplified by use of the Fourier transform (see, e.g., ref. 1). We present the explicit formulas for the collision integrals. Setting

$$g_2(\theta;m_1,m_2)=\pi^2\left[2\,\frac{\varepsilon(m_1,m_2)(m_1+m_2)}{m_1m_2}\right]^{1/2} g_2(\theta)$$

$$g_2(\theta)=\sum_{k=1}^{\infty}\{[((2k-1)\,\pi+\theta)^2-\pi^2]^{-3/2}$$
$$+[((2k+1)-\theta)^2-\pi^2]^{-3/2}\}$$

$$p_2(m_1,m_2)=2\left[2\,\frac{\varepsilon(m_1,m_2)(m_1+m_2)}{m_1m_2}\right]^{1/2}$$

the collision terms in (3.7) become

$$I_B=\int_0^\infty dm_1\int_{\mathbb{R}^2} dw\int_{-\pi}^{+\pi} d\theta\; g_2(|\theta|;m_1,m_2)$$
$$\times\{f(m,v')\,f(m_1,w')-f(m,v)\,f(m_1,w)\}$$

$$I_{Sm}=\frac{1}{2}\int_0^m ds\int_{\mathbb{R}^2} dU\,p_2(s,m-s)\,f\left(s,v+\frac{s}{m}U\right)f\left(m-s,v-\frac{m-s}{m}U\right)$$

$$-f(m,v)\int_0^\infty ds\int_{\mathbb{R}^2} dw\;p_2(m,s)\,f(s,w)$$

with v' and w' given as in (3.8), provided that $v, w\in\mathbb{R}^2$ and $n=\{\cos\theta,\sin\theta\}$, $-\pi<\theta<\pi$.

We finally remark that for the potential (1.15) the Smoluchowski integral remains exactly the same as for the potential (3.1), i.e., it is given by the formulas (3.11–12). For this potential, the Boltzmann collision term has to be calculated as described in Section 2 from formula (1.16) with a proper regularization for $\rho \to \infty$. If ε in (1.15) is sufficiently small, the contribution of the second term in elastic collsions should be negligible, and the standard Landau–Fokker–Planck collision integral for pure Coulomb interactions should emerge. However, in some situations the Smoluchowski collision term should be important even when ε is small, because new qualitative effects like cluster formation are possible.

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